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Note about integrability and gauge fixing for bosonic string on $AdS_5 \times S^5$

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Abstract: This short note is devoted to the study of the integrability of the bosonic string on $AdS_5 \times S^5$ in the uniform light-cone gauge. We construct Lax connection for gauge fixed theory and we argue that it is flat.

Keywords: AdS-CFT Correspondence, Bosonic Strings, Integrable Field Theories.

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1. Introduction and summary

It is well known that the string sigma model on $AdS_5 \times S^5$ is classically integrable [1].[1] More precisely, the authors [1] found a Lax formulation of the equations of motion for the classical Green-Schwarz superstring that leads to the existence of an infinite tower of conserved charges in the classical world-sheet theory. It is important to stress that this Lax formulation was derived for diffeomorphism invariant and κ symmetry invariant theory.

On the other hand it was shown recently in [5] that this fact does not quite coincide with the standard definition of integrability. Integrability in the standard sense requires not only the existence of a tower of conserved charges but also requires that these charges be in involution. In other words the conserved charges should Poisson commute with each other. The analysis presented in [5] explicitly demonstrated that for classical string moving on $R \times S^3$ submanifold of $AdS_5 \times S^5$ that the Poisson brackets of conserved charges are in involution. Further, in our recent paper [2] we performed the Hamiltonian analysis of the same model on the world-sheet with general metric. We showed that in case when either the diffeomorphism invariance of the world-sheet theory was preserved or the components of the metric were fixed while the gauge symmetries generated by Virasoro generators were not fixed the theory is integrable in the sense advocated in [5].

The situation becomes more involved in case when the gauge fixing functions depend on the phase space variables. An example of such a gauge is *uniform light-cone gauge* [24, 25].[2] As the modest contribution to the study of the integrability of the gauge fixed theory we would like to present arguments that further support the claim that the string theory in uniform light-cone gauge is integrable. We explicitly construct Lax connection for bosonic sting on $AdS_5 \times S^5$ in uniform light-cone gauge and we argue that this Lax connection is flat.[3] These arguments are based on the T-duality approach for the gauge fixing that was introduced in [28].

[1] For some works considering integrability of sigma model on $AdS_5 \times S^5$, see [2 − 22]

[2] For recent discussion of this gauge, see for example [26, 27].

[3] For some previous works discussing the integrability of the gauge fixed theory, see [8, 13, 15].

More precisely, we start with the bosonic string on $AdS_5 \times S^5$, following formulation presented in [11, 15]. Our goal is to study the question whether the theory [15, 24, 25, 27] formulated in the uniform light-cone gauge is integrable as well. We proceed in following way. In order to find the formulation of the theory in the uniform light-cone gauge we use the approach presented in [26, 28] that is more convenient for the study of the gauge fixed theory. On the other hand we argue, following [11, 15], that due to the fact that the original Lax connection is not T-duality invariant we have to perform field redefinition that introduces new Lax connection that is T-duality invariant. Using this improved Lax connection we can define the Lax connection in T-dual background when we use the map between original and T-dual variables. As the next step we perform the gauge fixing following [26, 28]. Then we argue that the gauge fixed theory possesses the Lax connection that is flat.

The extension of this work is as follows. It is straightforward to apply an approach presented in this paper to the case of the full Green-Schwarz superstring, following very nice analysis presented in [7]. On the other hand the second extension of this work is more involved. Even if we were able to find Lax connection for gauge fixed theory the Poisson bracket of the spatial components of Lax connection has not been determined yet. While the calculation of the Poisson bracket between spatial components of Lax connection is straightforward [2, 5] in case of the gauge fixed action it is much more difficult [8]. Moreover, the Poisson bracket derived there does not seem to have the form presented in [38, 39]. While an existence of Lax connection for gauge fixed theory implies an existence of the infinite number of conserved charges the fact that the Poisson bracket of Lax connection [8] does not take the standard form implies that it is not clear that these charges are in involution. Clearly this issue deserves further study.

The organisation of this paper is as follows. In the next section (2) we introduce the principal chiral model that defines bosonic string on $AdS_5 \times S^5$. We define Lax connection that is invariant under T-duality and then we find Lax connection for the theory fixed in uniform light-cone gauge. This is the main result of our paper. On the other hand in order to have paper self-contained we include some well known materials to two appendices. Explicitly, in appendix A we review the derivation of T-duality rules for sigma model. Then in appendix B we present similar calculation in case of principal chiral model defined on group manifold. Explicitly, we show how T-duality is implemented in case of principal chiral model and its relation to integrability,[4] following [29 – 32]. We argue that for some special examples of principal chiral models T-dual models are also integrable. Unfortunately we are not able to answer the question of integrability of T-dual of principal chiral model in the full general case.

2. Integrability of gauge fixed bosonic string on $AdS_5 \times S^5$

The motivation for the study of the question whether the integrability of the principal chiral model is preserved under T-duality was to understand the integrability of the gauge

[4]For some reviews of T-duality, see [33 – 35].

fixed action for string on $AdS_5 \times S^5$. The problem is that this principal model does not have such a simple form as an example given in the end of the previous section and hence we have to proceed in different way.

Explicitly, let us consider action for bosonic string on $AdS_5 \times S^5$ in the form

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \sqrt{-\gamma}\gamma^{\alpha\beta} g_{MN} \partial_\alpha x^M \partial_\beta x^N \, , \tag{2.1}$$

where g_{MN} are metric components of $AdS_5 \times S^5$ whose explicit form is given below and where x^M label coordinates of this space.

In order to study the integrability properties of the theory we use the fact that we can write the sigma model action (2.1) as [15]

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \sqrt{-\gamma}\gamma^{\alpha\beta} \mathrm{Tr}(J_\alpha J_\beta) \, , \tag{2.2}$$

where

$$J_\alpha = G^{-1}\partial_\alpha G \, , \quad G = \begin{pmatrix} g_a & 0 \\ 0 & g_s \end{pmatrix} \, . \tag{2.3}$$

Here g_a and g_s are following 4×4 matrices

$$g_a = \begin{pmatrix} 0 & \mathcal{Z}_3 & -\mathcal{Z}_2 & \mathcal{Z}_1^* \\ -\mathcal{Z}_3 & 0 & \mathcal{Z}_1 & \mathcal{Z}_2^* \\ \mathcal{Z}_2 & -\mathcal{Z}_1 & 0 & -\mathcal{Z}_3^* \\ -\mathcal{Z}_1^* & -\mathcal{Z}_2^* & \mathcal{Z}_3^* & 0 \end{pmatrix} , \quad g_s = \begin{pmatrix} 0 & \mathcal{Y}_1 & -\mathcal{Y}_2 & \mathcal{Y}_3^* \\ -\mathcal{Y}_1 & 0 & \mathcal{Y}_3 & \mathcal{Y}_2^* \\ \mathcal{Y}_2 & -\mathcal{Y}_3 & 0 & \mathcal{Y}_1^* \\ -\mathcal{Y}_3^* & -\mathcal{Y}_2^* & -\mathcal{Y}_1^* & 0 \end{pmatrix} , \tag{2.4}$$

where $\mathcal{Z}_k, k = 1, 2, 3$ are the complex embedding coordinates for AdS_5 and $\mathcal{Y}_k \, , k = 1, 2, 3$ are the complex embedding coordinates for sphere. The matrix g_a is an element of the group $\mathrm{SU}(2, 2)$ since it can be shown that

$$g_a^\dagger E g_a = E \, , \quad E = \mathrm{diag}(-1, -1, 1, 1) \tag{2.5}$$

provided the following condition is satisfied

$$\mathcal{Z}_1^*\mathcal{Z}_1 + \mathcal{Z}_2^*\mathcal{Z}_2 - \mathcal{Z}_3^*\mathcal{Z}_3 = -1 \, . \tag{2.6}$$

In fact g_a describes embedding of an element of the coset space $\mathrm{SO}(4, 2)/\mathrm{SO}(5, 1)$ into group $\mathrm{SU}(2, 2)$ that is locally isomorphic to $\mathrm{SO}(4, 2)$. We use this isometry to work with 4×4 matrices rather with 6×6 ones. Note that due to the explicit choice of the coset representative above there is not any gauge symmetry left. Quite analogously g_s is unitary

$$g_s g_s^\dagger = 1 \tag{2.7}$$

on condition that $\mathcal{Y}_1^*\mathcal{Y}_1 + \mathcal{Y}_2^*\mathcal{Y}_2 + \mathcal{Y}_3^*\mathcal{Y}_3 = 1$. The matrix g_s describes an embedding of an element of the coset $\mathrm{SO}(6)/\mathrm{SO}(5)$ into $\mathrm{SU}(4)$ being isomorphic to $\mathrm{SO}(6)$.

The variables $\mathcal{Z}, \mathcal{Y}$ are related to the variables used in (2.1) as follows. The five sphere S^5 is parametrised by five variables: coordinates $y^i \, , i = 1, \ldots, 4$ and the angle variable

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ϕ. In terms of six real embedding coordinates $Y^A, A = 1, \dots, 6$ obeying the condition $Y_A Y^A = 1$ the parametrisation reads

$$\mathcal{Y}_1 = Y_1 + iY_2 = \frac{y_1 + iy_2}{1 + \frac{y^2}{4}} , \quad \mathcal{Y}_2 = Y_3 + iY_4 = \frac{y_3 + iy_4}{1 + \frac{y^2}{4}} ,$$
$$\mathcal{Y}_3 = Y_5 + iY_6 = \frac{1 - \frac{y^2}{4}}{1 + \frac{y^2}{4}} \exp(i\phi) . \tag{2.8}$$

In the same way we describe the AdS_5 space when we introduce four coordinates z_i and t. The embedding coordinates Z_A that obey $Z_A Z_B \eta^{AB} = -1$ with the metric $\eta^{AB} = (-1, 1, 1, 1, 1, -1)$ is now parametrised as

$$\mathcal{Z}_1 = Z_1 + iZ_2 = -\frac{z_1 + iz_2}{1 - \frac{z^2}{4}} , \qquad \mathcal{Z}_2 = Z_3 + iZ_4 = -\frac{z_3 + iz_4}{1 - \frac{z^2}{4}} ,$$
$$\mathcal{Z}_3 = Z_0 + iZ_5 = \frac{1 + \frac{z^2}{4}}{1 - \frac{z^2}{4}} \exp(it) . \tag{2.9}$$

Note that the line element for $AdS_5 \times S^5$ takes the form

$$ds^2 = -\frac{(1 + \frac{z^2}{4})^2}{(1 - \frac{z^2}{4})^2} dt^2 + \frac{1}{(1 - \frac{z^2}{4})^2} dz_i dz_i + \left(\frac{1 - \frac{y^2}{4}}{1 + \frac{y^2}{4}} \right)^2 d\phi^2 + \frac{1}{(1 + \frac{y^2}{4})^2} dy_i dy_i . \tag{2.10}$$

Now using the fact that the bosonic string on $AdS_5 \times S^5$ can be written as principal chiral model immediately implies an existence of the Lax connection

$$L_\alpha = \frac{1}{1 - \Lambda^2} (J_\alpha - \Lambda \gamma_{\alpha\beta} \epsilon^{\beta\gamma} J_\gamma) \tag{2.11}$$

that obeys the flatness condition

$$\partial_\alpha L_\beta - \partial_\beta L_\alpha + [L_\alpha, L_\beta] = 0 . \tag{2.12}$$

Note that $\gamma_{\alpha\beta}$ in (2.11) is general world-sheet metric and Λ is a spectral parameter.

Then it was shown in [2] that the Poisson brackets of spatial components of Lax connection implies an existence of infinite number of conserved charges that are in involution.

On the other hand it would be interesting to study the gauge fixed form of the theory and whether the integrability is preserved in this case. In fact it was shown in [13, 15] that for some form of the gauge fixing the theory is integrable as well. Now we would like to give an alternative argument that supports the integrability of the gauge fixed theory in uniform light-cone gauge.

Our approach is based on the definition of the gauge fixing introduced in [24, 26, 28]. Let us introduce following combinations

$$x^+ = (1 - a)t + a\phi , \qquad x^- = \phi - t ,$$
$$t = x^+ - ax^- , \qquad \phi = x^+ + (1 - a)x^- , \tag{2.13}$$

where a is a free parameter from interval $a \in [0, 1)$. Using these variables the action (2.1) takes the form

$$\begin{aligned} S = & -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \sqrt{-\gamma}\gamma^{\alpha\beta}(g_{++}\partial_\alpha x^+ \partial_\beta x^+ + 2g_{+-}\partial_\alpha x^+ \partial_\beta x^- + \\ & + g_{--}\partial_\alpha x^- \partial_\beta x^- + g_{mn}\partial_\alpha x^m \partial_\beta x^n) \,, \end{aligned} \tag{2.14}$$

where now

$$g_{++} = g_{tt} + g_{\phi\phi} \,, \quad g_{+-} = -a g_{tt} + (1-a) g_{\phi\phi} \,, \quad g_{--} = g_{tt} a^2 + (1-a)^2 g_{\phi\phi} \,. \tag{2.15}$$

and $x^m = (y_i, z_i)$. As the next step we perform T-duality along x^-. Using (A.9) we obtain the relation between original and T-dual variables in the form

$$\epsilon^{\alpha\beta}\partial_\beta \tilde{x}^- = \gamma^{\alpha\beta}\partial_\beta x^+ g_{+-} + \gamma^{\alpha\beta}\partial_\beta x^- g_{--} \,, \quad \tilde{x}^m = x^m \,, \quad \tilde{x}^+ = x^+ \,. \tag{2.16}$$

Then (2.16) also implies

$$\partial_\alpha x^- = -\frac{1}{g_{--}}(\partial_\alpha \tilde{x}^+ g_{+-} + \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma \tilde{x}^-) \,. \tag{2.17}$$

Note that formula's (A.11) also imply following forms of metric an two-form field components in T-dual theory

$$\begin{aligned} \tilde{g}_{--} &= \frac{1}{g_{--}} = \frac{1}{g_{tt}a^2 + (1-a)^2 g_{\phi\phi}} \,, \quad \tilde{g}_{+-} = 0 \,, \\ \tilde{g}_{++} &= g_{++} - \frac{g_{-+}^2}{g_{--}} = \frac{g_{tt} g_{\phi\phi}}{g_{tt}a^2 + (1-a)^2 g_{\phi\phi}} \,, \\ \tilde{g}_{mn} &= g_{mn} \,, \quad \tilde{b}_{-+} = \frac{g_{-+}}{g_{--}} = -\tilde{b}_{+-} = \frac{-a g_{tt} + (1-a) g_{\phi\phi}}{g_{tt}a^2 + (1-a)^2 g_{\phi\phi}} \,. \end{aligned} \tag{2.18}$$

As the next step we integrate out the world-sheet metric $\gamma_{\alpha\beta}$ and we obtain

$$\gamma_{\alpha\beta} = \partial_\alpha \tilde{x}^M \partial_\beta \tilde{x}^N \tilde{g}_{MN} \,. \tag{2.19}$$

Inserting this result to the T-dual action we obtain

$$S = -\frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \left[\sqrt{-\det \tilde{g}_{MN}\partial_\alpha \tilde{x}^M \partial_\beta \tilde{x}^N} + \frac{1}{2}\varepsilon^{\alpha\beta}\tilde{b}_{MN}\partial_\alpha \tilde{x}^M \partial_\beta \tilde{x}^N \right] \,.$$

Finally, the uniform gauge fixing is achieved as [26]

$$\tilde{x}^+ = \frac{\tau}{1-a} \,, \quad \tilde{\phi} = \frac{J_+ \sigma}{2\pi} \,. \tag{2.20}$$

However since this approach is based on T-duality transformation of the action we come to the puzzle since the Lax connection explicitly depends on the variables that parametrise isometry directions. To resolve this problem we follow [11, 15].

We start with the original form of the action (2.2) with general world-sheet metric. Then we use the fact that matrices g_s, g_a enjoy following property [11, 15]

$$\begin{aligned} g_s(y,\phi) &= M(\phi)\hat{g}_s(y)M(\phi)\,, \\ g_a(z,t) &= N(t)\hat{g}_a(z)N(t)\,, \end{aligned} \tag{2.21}$$

where

$$M(\phi) = \begin{pmatrix} e^{-\frac{i}{2}\phi} & 0 & 0 & 0 \\ 0 & e^{\frac{i}{2}\phi} & 0 & 0 \\ 0 & 0 & e^{\frac{i}{2}\phi} & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{2}\phi} \end{pmatrix}\,, \quad N(t) = \begin{pmatrix} e^{\frac{i}{2}t} & 0 & 0 & 0 \\ 0 & e^{\frac{i}{2}t} & 0 & 0 \\ 0 & 0 & e^{-\frac{i}{2}t} & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{2}t} \end{pmatrix}\,, \tag{2.22}$$

and

$$\hat{g}_a = \begin{pmatrix} 0 & \frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}} & -\mathcal{Z}_2 & \mathcal{Z}_1^* \\ -\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}} & 0 & \mathcal{Z}_1 & \mathcal{Z}_2^* \\ \mathcal{Z}_2 & -\mathcal{Z}_1 & 0 & -\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}} \\ -\mathcal{Z}_1^* & -\mathcal{Z}_2^* & \frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}} & 0 \end{pmatrix}\,, \quad \hat{g}_s = \begin{pmatrix} 0 & \mathcal{Y}_1 & -\mathcal{Y}_2 & \frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}} \\ -\mathcal{Y}_1 & 0 & \frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}} & \mathcal{Y}_2^* \\ \mathcal{Y}_2 & -\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}} & 0 & \mathcal{Y}_1^* \\ -\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}} & -\mathcal{Y}_2^* & -\mathcal{Y}_1^* & 0 \end{pmatrix}\,. \tag{2.23}$$

Note that in this case the matrix G can be written as

$$G = \mathbf{M}\hat{G}\mathbf{M}\,, \quad \mathbf{M} = \begin{pmatrix} N(t) & 0 \\ 0 & M(\phi) \end{pmatrix}\,, \quad \hat{G} = \begin{pmatrix} \hat{g}_a & 0 \\ 0 & \hat{g}_s \end{pmatrix}\,. \tag{2.24}$$

Using this factorisation property we obtain

$$J = G^{-1}dG = \mathbf{M}^{-1}\left(\hat{G}^{-1}d\hat{G} + \frac{i}{2}\hat{G}^{-1}d\mathbf{\Phi}\hat{G} + \frac{i}{2}d\mathbf{\Phi}\right)\mathbf{M} \equiv \mathbf{M}^{-1}\hat{J}\mathbf{M}\,, \tag{2.25}$$

where

$$\mathbf{\Phi} = \begin{pmatrix} \Phi & 0 \\ 0 & \Psi \end{pmatrix}\,, \tag{2.26}$$

and where $\Phi = \mathrm{diag}(-\phi, \phi, \phi, -\phi)$ and $\Psi = \mathrm{diag}(t, t, -t, -t)$. Now using (2.25) we define Lax connection $\hat{L}$ from the original one (2.11) as

$$L_\alpha = \mathbf{M}^{-1}\hat{L}_\alpha\mathbf{M} \tag{2.27}$$

Then the flatness condition (2.12) implies

$$\begin{aligned} \partial_\alpha L_\beta - \partial_\beta L_\alpha + [L_\alpha, L_\beta] = \mathbf{M}^{-1}\Bigg(&\partial_\alpha\left(\hat{L}_\beta - \frac{i}{2}\partial_\beta\mathbf{\Phi}\right) - \partial_\beta\left(\hat{L}_\alpha - \frac{i}{2}\partial_\alpha\mathbf{\Phi}\right) + \\ &+ \left[\left(\hat{L}_\alpha - \frac{i}{2}\partial_\alpha\mathbf{\Phi}\right), \left(\hat{L}_\beta - \frac{i}{2}\partial_\beta\mathbf{\Phi}\right)\right]\Bigg)\mathbf{M} = 0 \end{aligned} \tag{2.28}$$

Hence we see that instead of the original Lax connection we can find another one that is again flat

$$\mathbf{L}_\alpha = \hat{L}_\alpha(\mathbf{\Phi}) - \frac{i}{2}\partial_\alpha\mathbf{\Phi} = \frac{1}{1-\Lambda^2}(\hat{J}_\alpha(\mathbf{\Phi}) - \Lambda\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\hat{J}_\gamma(\mathbf{\Phi})) - \frac{i}{2}\partial_\alpha\mathbf{\Phi}\,, \tag{2.29}$$

where we explicitly stressed the dependence of $\hat{J}$ on $\mathbf{\Phi}$ as follows from (2.25). The advantage of the Lax connection $\mathbf{L}$ is that it now depends on a derivative of $\mathbf{\Phi}$ only. This result implies that the Lax connection $\mathbf{L}$ is useful for the definition of the Lax connection for T-dual theory. Further, since the relations between original and T-dual variables are valid on-shell we obtain that the Lax connection defined using the T-dual variables is flat as well. More precisely, using (2.13) we replace t and ϕ in Φ,Ψ with x^+,x^- so that

$$\begin{aligned}\Phi &= (x^+ + (1-a)x^-)\Omega\,, \quad \Omega = \mathrm{diag}(-1,1,1,-1)\,,\\ \Psi &= (x^+ - ax^-)\Sigma\,, \quad \Sigma = \mathrm{diag}(1,1,-1,-1)\,.\end{aligned} \tag{2.30}$$

Then using the relations between original and T-dual variables (2.16) we find

$$\begin{aligned}\partial_\alpha\Phi &= \left[\partial_\alpha\tilde{x}^+ - (1-a)\frac{1}{g_{--}}(\partial_\alpha\tilde{x}^+ g_{+-} + \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\tilde{x}^-)\right]\Omega\,,\\ \partial_\alpha\Psi &= \left[\partial_\alpha\tilde{x}^+ + a\frac{1}{g_{--}}(\partial_\alpha\tilde{x}^+ g_{+-} + \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\tilde{x}^-)\right]\Sigma\,.\end{aligned} \tag{2.31}$$

Then we can define Lax connection for T-dual theory in the form

$$\mathbf{L}_\alpha = \frac{1}{1-\Lambda^2}(\hat{J}_\alpha(\mathbf{\Phi}) - \Lambda\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\hat{J}_\gamma(\mathbf{\Phi})) - \frac{i}{2}\partial_\alpha\mathbf{\Phi}\,, \tag{2.32}$$

where now $\mathbf{\Phi}$ depends on $\tilde{x}^M$ through the relations (2.31). Since (2.31) hold on-shell the Lax connection defined in T-dual theory (2.32) is flat as well. Note that we still presume that the world-sheet metric $\gamma_{\alpha\beta}$ given in (2.32) is general. However as the next step in the gauge fixing procedure we integrate out it and we get (2.19). Again, since Lax connection is flat for any metric it is flat for metric that is on-shell (2.19). Finally, we perform the gauge fixing when we insert (2.20) into (2.31) and we obtain components $\partial\mathbf{\Phi}$ for gauge fixed theory

$$\begin{aligned}\partial_\tau\Phi &= \left[\frac{1}{1-a} - (1-a)\frac{J_+}{2\pi}\frac{\gamma_{\tau\tau}}{g_{--}\sqrt{-\gamma}}\right]\Omega\,,\\ \partial_\sigma\Phi &= \frac{(1-a)J_+}{2\pi}\frac{\gamma_{\sigma\tau}}{\sqrt{-\gamma}g_{--}}\Omega\,,\\ \partial_\tau\Psi &= \left[\frac{1}{1-a} + \frac{aJ_+}{2\pi}\frac{\gamma_{\tau\tau}}{g_{--}\sqrt{-\gamma}}\right]\Sigma\,.\\ \partial_\sigma\Psi &= -\frac{aJ_+}{2\pi}\frac{\gamma_{\sigma\tau}}{\sqrt{-\gamma}g_{--}}\Sigma\,,\end{aligned} \tag{2.33}$$

where now

$$\begin{aligned}\gamma_{\tau\tau} &= \frac{1}{(1-a^2)}\frac{g_{tt}g_{\phi\phi}}{g_{tt}a^2 + (1-a)^2 g_{\phi\phi}} + g_{mn}\partial_\tau x^m\partial_\tau x^n\,,\\ \gamma_{\tau\sigma} &= g_{mn}\partial_\tau x^m\partial_\sigma x^n\,,\\ \gamma_{\sigma\sigma} &= \frac{J_+^2}{4\pi^2(g_{tt}a^2 - (1-a)^2 g_{\phi\phi})} + g_{mn}\partial_\sigma x^m\partial_\sigma x^n\,.\end{aligned} \tag{2.34}$$

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As it is clear from arguments given above the Lax connection for theory in the uniform light-cone gauge (2.20) is flat. We mean the study of the integrability of the gauge fixed theory in the Lagrange formalism can be considered as an useful alternative to the analysis presented in [13, 15].

A. T-duality for sigma model

In this appendix we review we introduce standard notation considering T-duality. We start with the sigma model action that describes the propagation of closed string on the background with several U(1) isometries

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \sqrt{-\gamma}[\gamma^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^j g_{ij} - \epsilon^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^j b_{ij} + \\ + 2\partial_\alpha\phi^i(\gamma^{\alpha\beta}u_{\beta,i} - \epsilon^{\alpha\beta}v_{\beta,i}) + \mathcal{L}_{rest}] \ . \tag{A.1}$$

As usual we have introduced the effective string tension $\frac{\sqrt{\lambda}}{2\pi}$ that is identified with the 't Hooft coupling in the AdS/CFT correspondence, $\gamma_{\alpha\beta}$ is worldsheet metric with Minkowski signature that in conformal gauge is $\gamma^{\alpha\beta} = (-1, 1)$ and $\epsilon^{\alpha\beta} = \frac{\varepsilon^{\alpha\beta}}{\sqrt{-\gamma}}, \varepsilon^{\tau\sigma} = -\varepsilon^{\sigma\tau} = 1$. Next we assume that the action is invariant under the U(1) isometry transformations that are geometrically realised as shifts of the angle variables $\phi^i, i = 1, 2, \ldots, d$. In other words the string background contains the d-dimensional torus T^d. The action (A.1) explicitly shows the dependence on ϕ^i and their coupling to the background fields g_{ij}, b_{ij} and $u_{\alpha,i}, v_{\alpha,i}$. These background fields are independent on ϕ^i but can depend on other bosonic string coordinates which are neutral under the U(1) isometry transformations. Finally $\mathcal{L}_{rest}$ denotes the part of the Lagrangian that depends on other fields of the theory.

As previous discussion suggests the action (A.1) is invariant under the constant shift of ϕ^i

$$\phi'^i(\tau, \sigma) = \phi^i(\tau, \sigma) + \epsilon^i \ . \tag{A.2}$$

Corresponding Noether currents have the form

$$J_i^\alpha = -\frac{\sqrt{\lambda}}{2\pi}\sqrt{-\gamma}(\gamma^{\alpha\beta}\partial_\beta\phi^j g_{ji} - \epsilon^{\alpha\beta}\partial_\beta\phi^j b_{ij} + \gamma^{\alpha\beta}u_{\beta,i} - \epsilon^{\alpha\beta}v_{\beta,i}) \tag{A.3}$$

and obeys the equation

$$\partial_\alpha J_i^\alpha = 0 \tag{A.4}$$

as a consequence of the equations of motion.

Now we are ready to study T-duality for this model. We closely follow [35]. Let us start with the T-duality on a circle parametrised by ϕ^1. As the next step we gauge the shift symmetry $\phi'^1 = \phi^1 + \epsilon^1$ so that ϵ^1 is now function of τ, σ. If we require that the action is invariant under the non-constant transformation we have to introduce the appropriate gauge field A_α in such a way that

$$\partial_\alpha\phi^1 \to (\partial_\alpha\phi^1 + A_\alpha) \equiv D_\alpha\phi^1 \ . \tag{A.5}$$

At the same time we add to the action the term $\tilde{\phi}^1\epsilon^{\alpha\beta}F_{\alpha\beta}$ in order to assure that the gauge field has trivial dynamics. The field $\tilde{\phi}^1$ is corresponding Lagrange multiplier. Then we obtain the gauge invariant action

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\sqrt{-\gamma}[&\gamma^{\alpha\beta}D_\alpha\phi^1 D_\beta\phi^1 g_{11} + 2\gamma^{\alpha\beta}D_\alpha\phi^1\partial_\beta\phi^a g_{1a} + \gamma^{\alpha\beta}\partial_\alpha\phi^a\partial_\beta\phi^b g_{ab} - \\ &-\epsilon^{\alpha\beta}\partial_\alpha\phi^a\partial_\beta\phi^b b_{ab} - 2\epsilon^{\alpha\beta}D_\alpha\phi^1\partial_\beta\phi^b b_{1b} + \\ &+2D_\alpha\phi^1(\gamma^{\alpha\beta}u_{\beta,1} - \epsilon^{\alpha\beta}v_{\beta,1}) + 2\partial_\alpha\phi^a(\gamma^{\alpha\beta}u_{\beta,a} - \epsilon^{\alpha\beta}v_{\beta,a}) + \tilde{\phi}^1\epsilon^{\alpha\beta}F_{\alpha\beta} + \mathcal{L}_{rest}]\,, \end{aligned} \tag{A.6}$$

where $a,b = 2,\dots,d$. Now thanks to the gauge invariance we can fix the gauge $\phi^1 = 0$ so that the action above takes the form

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\sqrt{-\gamma}[&\gamma^{\alpha\beta}A_\alpha A_\beta g_{11} + 2\gamma^{\alpha\beta}A_\alpha\partial_\beta\phi^a g_{1a} + \gamma^{\alpha\beta}\partial_\alpha\phi^a\partial_\beta\phi^b g_{ab} - \\ &-\epsilon^{\alpha\beta}\partial_\alpha\phi^a\partial_\beta\phi^b b_{ab} - 2\epsilon^{\alpha\beta}A_\alpha\partial_\beta\phi^b b_{1b} + \\ &+2A_\alpha(\gamma^{\alpha\beta}u_{\beta,1} - \epsilon^{\alpha\beta}v_{\beta,1}) + 2\partial_\alpha\phi^a(\gamma^{\alpha\beta}u_{\beta,a} - \epsilon^{\alpha\beta}v_{\beta,a}) + \tilde{\phi}^1\epsilon^{\alpha\beta}F_{\alpha\beta} + \mathcal{L}_{rest}]\,. \end{aligned} \tag{A.7}$$

If we now integrate $\tilde{\phi}^1$ we obtain that $F_{\alpha\beta} = 0$ and hence $A_\alpha = \partial_\alpha\theta$. Inserting back to the action (A.7) we obtain the original action (A.1) after identification $\theta = \phi^1$. On the other hand if we integrate out A_α we obtain

$$A_\alpha = \frac{1}{g_{11}}(-\partial_\alpha\phi^a g_{1a} + \gamma_{\alpha\beta}\epsilon^{\beta\rho}\partial_\rho\phi^a b_{1a} - (u_{\alpha,1} - \gamma_{\alpha\beta}\epsilon^{\beta\rho}v_{\rho,1}) - \gamma_{\alpha\beta}\epsilon^{\beta\rho}\partial_\rho\tilde{\phi}^1)\,. \tag{A.8}$$

Since we have argued that A_α can be related to the original coordinate ϕ^1 as $A_\alpha = \partial_\alpha\phi^1$ the relation (A.8) implies following relation between original and T-dual variables ϕ^i and $\tilde{\phi}^i$

$$\begin{aligned} \epsilon^{\alpha\rho}\partial_\rho\tilde{\phi}^1 &= -\gamma^{\alpha\rho}g_{11}\partial_\rho\phi^1 - \gamma^{\alpha\rho}\partial_\rho\phi^a g_{1a} + \epsilon^{\alpha\rho}\partial_\rho\phi^a b_{1a} - \gamma^{\alpha\rho}u_{\rho,1} + \epsilon^{\alpha\rho}v_{\rho,1}\,, \\ \tilde{\phi}^a &= \phi^a\,. \end{aligned} \tag{A.9}$$

Now plugging the result (A.8) into the action above we obtain the action equivalent to (A.1)

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\sqrt{-\gamma}[&\gamma^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j\tilde{g}_{ij} - \epsilon^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j\tilde{b}_{ij} + \\ &+2\partial_\alpha\phi^i(\gamma^{\alpha\beta}\tilde{u}_{\beta,i} - \epsilon^{\alpha\beta}\tilde{v}_{\beta,i}) + \tilde{\mathcal{L}}_{rest}]\,, \end{aligned} \tag{A.10}$$

where [36, 37]

$$\begin{aligned} \tilde{g}_{11} &= \frac{1}{g_{11}}\,, \qquad \tilde{g}_{ab} = g_{ab} - \frac{g_{a1}g_{1b} - b_{1a}b_{1b}}{g_{11}}\,, \qquad \tilde{g}_{1a} = \frac{b_{1a}}{g_{11}}\,, \\ \tilde{b}_{ab} &= b_{ab} - \frac{g_{1a}b_{1b} - b_{1a}g_{1b}}{g_{11}}\,, \qquad \tilde{b}_{1a} = \frac{g_{1a}}{g_{11}}\,, \qquad \tilde{b}_{a1} = -\frac{g_{1a}}{g_{11}}\,, \\ \tilde{u}_{\alpha,1} &= \frac{v_{\alpha,1}}{g_{11}}\,, \qquad \tilde{v}_{\alpha,1} = \frac{u_{\alpha,1}}{g_{11}}\,, \\ \tilde{u}_{\alpha,a} &= u_{\alpha,a} - \frac{g_{1a}u_{\beta,1} - b_{1a}v_{\alpha,1}}{g_{11}}\,, \\ \tilde{v}_{\alpha,a} &= v_{\alpha,a} - \frac{g_{1a}v_{\alpha,1} - b_{1a}u_{\alpha,1}}{g_{11}}\,, \\ \tilde{\mathcal{L}}_{rest} &= \mathcal{L}_{rest} - \gamma^{\alpha\beta}\frac{u_{\alpha,1}u_{\beta,1} - v_{\alpha,1}v_{\beta,1}}{g_{11}} + \epsilon^{\alpha\beta}\frac{u_{\alpha,1}v_{\beta,1} - v_{\alpha,1}u_{\beta,1}}{g_{11}}\,. \end{aligned} \tag{A.11}$$

These relations will be useful when we discuss the gauge fixed form of the bosonic string on $AdS_5 \times S^5$ in section (B). On the other hand in the next section we perform the same T-duality analysis for the special case of the sigma model that can be written in the form of principal chiral model.

B. T-duality for principal chiral model and integrability

Let us consider the special case of the sigma model action (A.1) that is known as principal chiral model

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \sqrt{-\gamma}\gamma^{\alpha\beta} K_{AB} J^A_\alpha J^B_\beta \,, \tag{B.1}$$

where

$$J = G^{-1}dG = J^A T_A \,, \tag{B.2}$$

and where G is a group element from the group $\mathcal{G}$ and where T_A are generators of corresponding algebra $\mathbf{g}$ that obey following relations

$$[T_A, T_B] = f^C_{AB} T_C \,, \quad \mathrm{Tr}(T_A T_B) = K_{AB} \,, \tag{B.3}$$

where K_{AB} is invertible matrix and where $f^C_{AB} = -f^C_{BA}$ are structure constants of the algebra $\mathbf{g}$. The indices A, B label components of the basis T_A. If we parametrise the group element with the fields x^M we can write the current J^A_α as

$$J^A_\alpha = E^A_M \partial_\alpha x^M \ . \tag{B.4}$$

Finally we introduced the metric

$$g_{MN} = E^A_M K_{AB} E^B_N \tag{B.5}$$

defined on some target manifold labelled with coordinates x^M. In this interpretation E^A_M are vielbeins of the target manifold [40]. Note also that E^A can be written as

$$E^A = \mathrm{Tr}(G^{-1}dG T_B) K^{BA} \tag{B.6}$$

and hence the line element ds^2 can be written as

$$ds^2 = \mathrm{Tr}(G^{-1}dG G^{-1}dG) \ . \tag{B.7}$$

It is well known that the principal chiral model (B.1) is integrable [40]. More precisely, we can find Lax connection for the action (B.1) that is flat. Further, we can argue that this model possesses infinite number of integrals of motion that are in involution [2].

Following [29] we now consider the case when algebra $\mathbf{g}$ contains Cartan sub algebra

$$T_i \,, \quad [T_i, T_j] = 0 \,, \quad i = 1, \ldots, d \,, \quad \mathrm{Tr}(T_i T_j) = K_{ij} \,, \tag{B.8}$$

where d is the rank of the algebra. Let us also parametrise the group element as

$$G = e^{\sum_{i=1}^d \alpha^i T_i} h \ . \tag{B.9}$$

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Using (B.9) we obtain

$$\begin{aligned}\gamma^{\alpha\beta}\mathrm{Tr}(J_\alpha J_\beta) &= \gamma^{\alpha\beta}\mathrm{Tr}((h^{-1}\partial_\alpha\alpha^i T_i h + h^{-1}\partial_\alpha h)(h^{-1}\partial_\beta\alpha^j T_j h + h^{-1}\partial_\beta h)) = \\ &= \gamma^{\alpha\beta}\partial_\alpha\alpha^i K_{ij}\partial_\beta\alpha^j + 2\gamma^{\alpha\beta}\partial_\alpha\alpha^i H_{i\beta} + \gamma^{\alpha\beta}\mathrm{Tr}(h^{-1}\partial_\alpha h h^{-1}\partial_\beta h)\,,\end{aligned} \tag{B.10}$$

where

$$H_{i\alpha} \equiv \mathrm{Tr}(T_i \partial_\alpha h h^{-1})\ . \tag{B.11}$$

Now we are ready to study T-duality for this form of principal chiral model. In order to have contact with the discussion performed in next section let us consider slightly more general case. Explicitly, let us take first two α's and consider following combination

$$\alpha^\alpha = \Gamma^\alpha_y \gamma^y\,, \quad \alpha = 1,2\,, \quad x,y = 1,2\,, \tag{B.12}$$

where Γ^x_y are constant parameters. Using (B.12) the action (B.1) can be written as

$$\begin{aligned}S = -\frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\sqrt{-\gamma}[&\gamma^{\alpha\beta}\partial_\alpha\gamma^1\partial_\beta\gamma^1 K'_{11} + 2\gamma^{\alpha\beta}\partial_\alpha\gamma^1\partial_\beta\gamma^2 K'_{12} + \gamma^{\alpha\beta}\partial_\alpha\gamma^2\partial_\beta\gamma^2 K'_{22} + \\ &+\gamma^{\alpha\beta}\partial_\alpha\alpha^a\partial_\beta\alpha^b K_{ab} + 2\gamma^{\alpha\beta}\partial_\alpha\gamma^1 H'_{1\beta} + 2\gamma^{\alpha\beta}\partial_\alpha\gamma^2 H'_{2\beta} + \\ &+2\gamma^{\alpha\beta}\partial_\alpha\alpha^a H_{a\beta} + \gamma^{\alpha\beta}\mathrm{Tr}(h^{-1}\partial_\alpha h h^{-1}\partial_\beta h)]\,,\end{aligned} \tag{B.13}$$

where we also presumed that the metric K_{ij} is block diagonal so that $K_{\alpha a} = 0\,, a, b = 3, \ldots, d$. In (B.13) we also introduced the notation

$$K'_{xy} = \Gamma^\alpha_x \Gamma^\beta_y K_{\alpha\beta}\,, \quad H'_{x\alpha} = \Gamma^\beta_x H_{\beta\alpha}\ . \tag{B.14}$$

Let us now consider T-duality along the direction labelled with γ^1. As in the previous section we gauge the theory corresponding to the shift of γ^1

$$\partial_\alpha\gamma^1 \to D_\alpha\gamma^1 = \partial_\alpha\gamma^1 + A_\alpha\gamma^1\ . \tag{B.15}$$

Then if we fix the gauge with $\gamma^1 = 0$ the action takes the form

$$\begin{aligned}S = -\frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\sqrt{-\gamma}[&\gamma^{\alpha\beta}A_\alpha A_\beta K'_{11} + 2\gamma^{\alpha\beta}A_\alpha\partial_\beta\gamma^2 K'_{12} + \gamma^{\alpha\beta}\partial_\alpha\gamma^2\partial_\beta\gamma^2 K'_{22} + \\ &+\gamma^{\alpha\beta}\partial_\alpha\alpha^a\partial_\beta\alpha^b K_{ab} + 2\gamma^{\alpha\beta}A_\alpha H'_{1\beta} + \\ &+2\gamma^{\alpha\beta}\gamma^2 H'_{2\beta} + 2\gamma^{\alpha\beta}\partial_\alpha\alpha^a H_{a\beta} + \gamma^{\alpha\beta}\mathrm{Tr}(h^{-1}\partial_\alpha h h^{-1}\partial_\beta h) + \epsilon^{\alpha\beta}\tilde{\phi}F_{\alpha\beta}]\ .\end{aligned} \tag{B.16}$$

If we integrate $\tilde{\phi}$ we obtain $\epsilon^{\alpha\beta}F_{\alpha\beta} = 0$ that can be solved with

$$A_\alpha = \partial_\alpha\gamma^1 \tag{B.17}$$

and we recover the original action. On the other hand if we integrate A_α we obtain

$$A_\alpha = -\frac{1}{K'_{11}}[\partial_\alpha\gamma^1 K'_{21} + H'_{1\beta} + \gamma_{\alpha\gamma}\epsilon^{\gamma\beta}\partial_\beta\tilde{\phi}]\ . \tag{B.18}$$

Since $A_\alpha = \partial_\alpha \gamma^1$ this equation determines the relation between original and dual variables

$$\partial_\alpha \gamma^1 = -\frac{1}{K'_{11}}[\partial_\alpha \gamma^2 K'_{21} + H'_{1\beta} + \gamma_{\alpha\gamma}\epsilon^{\gamma\beta}\partial_\beta\tilde{\phi}] \ . \tag{B.19}$$

Inserting (B.19) into (B.16) we obtain dual action

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\sqrt{-\gamma}\Bigg[&\gamma^{\alpha\beta}\partial_\alpha\gamma^2\partial_\beta\gamma^2\left(K'_{22}-\frac{K'^2_{12}}{K'_{11}}\right)+\gamma^{\alpha\beta}\partial_\alpha\tilde{\phi}\partial_\beta\tilde{\phi}\frac{1}{K'_{11}}+\\ &+2\gamma^{\alpha\beta}\partial_\alpha\gamma^2\left(H'_{2\beta}-\frac{K'_{21}}{K'_{11}}H'_{1\beta}\right)+\gamma^{\alpha\beta}\partial_\alpha\alpha^a\partial_\beta\alpha^b K_{ab}+\\ &+2\gamma^{\alpha\beta}\partial_\alpha\alpha^a H_{a\beta}+\gamma^{\alpha\beta}\left(\mathrm{Tr}(h^{-1}\partial_\alpha h h^{-1}\partial_\beta h)-\frac{1}{K'_{11}}H'_{1\alpha}H'_{1\beta}\right)-\\ &-\partial_\alpha\gamma^2\partial_\beta\tilde{\phi}\epsilon^{\alpha\beta}\frac{2K'_{12}}{K'_{11}}-\epsilon^{\alpha\beta}\partial_\beta\tilde{\phi}\frac{2H'_{1\alpha}}{K'_{11}}\Bigg] \ . \end{aligned} \tag{B.20}$$

Let us now observe that we can write

$$\begin{aligned} H'_{x\alpha} &= \Gamma^\beta_x \mathrm{Tr}(h^{-1}T_\beta h h^{-1}\partial_\alpha h) = \Gamma^\beta_x \mathrm{Tr}(h^{-1}T_\beta h T_A)K^{AB}\mathrm{Tr}(T_B h^{-1}\partial_\alpha h) = \\ &= \Gamma^\beta_x E^A_\beta K_{AB} E^B_m \partial_\alpha x^m = \Gamma^\beta_x g_{\beta m}\partial_\alpha x^m \equiv g'_{xm}\partial_\alpha x^m \ , \\ K'_{xy} &= \Gamma^\alpha_x \Gamma^\beta_y \mathrm{Tr}(T_\alpha T_\beta) = \Gamma^\alpha_x\Gamma^\beta_y g_{\alpha\beta} \equiv g'_{xy} \end{aligned} \tag{B.21}$$

and consequently

$$\begin{aligned} \gamma^{\alpha\beta}\partial_\alpha\gamma^2\partial_\beta\gamma^2\left(K'_{22}-\frac{K'^2_{12}}{K'_{11}}\right) &= \gamma^{\alpha\beta}\partial_\alpha\gamma^2\partial_\beta\gamma^2\left(g'_{22}-\frac{g'_{12}g'_{12}}{g'_{11}}\right) \ , \\ \gamma^{\alpha\beta}\partial_\alpha\gamma^2\left(H'_{2\beta}-\frac{K'_{21}}{K'_{11}}H'_{1\beta}\right) &= \gamma^{\alpha\beta}\left(g'_{2m}-\frac{g'_{12}g'_{1m}}{g'_{11}}\right)\partial_\alpha\gamma^2\partial_\beta x^m \ , \\ \gamma^{\alpha\beta}\left(\mathrm{Tr}(h^{-1}\partial_\alpha h h^{-1}\partial_\beta h)-\frac{1}{K'_{11}}H'_{1\alpha}H'_{1\beta}\right) &= \gamma^{\alpha\beta}\left(g'_{mn}-\frac{g'_{1m}g'_{1n}}{g'_{11}}\right)\partial_\alpha x^m\partial_\beta x^n \ , \\ \partial_\alpha\gamma^2\partial_\beta\tilde{\phi}\epsilon^{\alpha\beta}\frac{2K'_{12}}{K'_{11}} &= \partial_\alpha\gamma^2\partial_\beta\tilde{\phi}\epsilon^{\alpha\beta}\frac{g'_{12}}{g'_{11}}-\partial_\alpha\tilde{\phi}\partial_\beta\gamma^2\epsilon^{\alpha\beta}\frac{g'_{12}}{g'_{11}} \ , \\ \epsilon^{\alpha\beta}\partial_\beta\tilde{\phi}\frac{2H'_{1\alpha}}{K'_{11}} &= \epsilon^{\alpha\beta}\partial_\alpha x^m\partial_\beta\tilde{\phi}\frac{g'_{1m}}{g'_{11}}-\epsilon^{\alpha\beta}\partial_\alpha\tilde{\phi}\partial_\beta x^m\frac{g'_{1m}}{g'_{11}} \ . \end{aligned} \tag{B.22}$$

In other words T-dual action (B.20) has exactly the same form as the action (A.10) with the metric and two form components given by Buscher's rules (A.11) when we replace g_{MN} with g'_{MN}.

Let us now restrict to the case when the metric g'_{MN} is diagonal and consequently $H'_{x\alpha} = 0$. Then, in the similar way as in [29] we introduce following generators of the sub algebra of Cartan algebra

$$\tilde{T}_2 = T'_2 - \frac{K'_{12}}{K'_{11}}T'_1 \ , \quad \tilde{T}_1 = \frac{1}{K'_{11}}T'_1 \ , \quad \tilde{T}_a = T_a \tag{B.23}$$

and consider following group element

$$\tilde{G} = e^{\tilde{\alpha}^i \tilde{T}_i} h \tag{B.24}$$

with corresponding current

$$\tilde{J} = \tilde{G}^{-1} d\tilde{G} = h^{-1} d\tilde{\alpha}^i \tilde{T}_i h + h^{-1} dh \ . \tag{B.25}$$

Then we can write T-dual action (B.20) in the form

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \sqrt{-\gamma} \left[\gamma^{\alpha\beta} \mathrm{Tr} \tilde{J}_\alpha \tilde{J}_\beta - \partial_\alpha \gamma^2 \partial_\beta \tilde{\phi} \epsilon^{\alpha\beta} \frac{2K'_{12}}{K'_{11}} \right] . \tag{B.26}$$

Note that the last term can be written as

$$\begin{aligned} \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \partial_\alpha \gamma^2 \partial_\beta \tilde{\phi} \varepsilon^{\alpha\beta} \frac{2K'_{12}}{K'_{11}} &= \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \partial_\alpha \left[\gamma^2 \varepsilon^{\alpha\beta} \partial_\beta \tilde{\phi} \frac{2K'_{12}}{K'_{11}} \right] - \\ &- \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \gamma^2 \partial_\alpha [\varepsilon^{\alpha\beta} \partial_\beta \tilde{\phi}] \frac{2K'_{12}}{K'_{11}} \end{aligned} \tag{B.27}$$

and hence does not affect the equations of motion. The first term is total derivative and can be discarded from the action and the second one vanishes due to the antisymmetry of $\varepsilon^{\alpha\beta}$.

The fact that T-dual action (B.26) has again form of the principal chiral model [5] implies that T-dual theory is integrable as well. On the other hand the form of the group element (B.9) is rather special. For example, the principal chiral model that describes bosonic string on $AdS_5 \times S^5$ does not have such a simple form.

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